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Rigorous bounds on the power spectrum of arbitrary prime η-order renormalisation group equations

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Abstract. We give rigorous bound inequalities for the universal scaling factors with their numerical values ($n = 3, 4, 5$), of the power spectrum for a renormalisation group equation of arbitrary prime order η by using a generalisation of the method of Collet-Eckmann-Thomas.

To date there have been both quantitative and qualitative studies of the power spectrum of a bifurcation sequence. First, Feigenbaum (1979, 1980) and Nauenberg and Rudnick (1981) approximately calculated the scaling factor of the power spectrum for the period-doubling sequence. Then Collet *et al* (1981), using the method of the renormalisation group equation, gave the lower and upper bounds on scaling factors by using rigorous qualitative analysis. However, attention has been restricted to the period-doubling sequence only. We will try to generalise these methods to arbitrary period-n-tupling sequences of higher order in multifurcation phenomena. Here our generalisation is with respect to the latter only (Collet *et al* 1981), but we published a generalisation of the former (Feigenbaum 1979, 1980) elsewhere (Peng *et al* 1985). In our studies we found that the power spectrum of multifurcation sequences has an important characteristic: their subcomponents have strongly coupled phase factors, while the odd subcomponents of a bifurcation sequence do not. These phase factors make the power spectrum of each subcomponent (ν) at the same level N form a finite automorphism group transformation. When the order η of the renormalisation group equation approaches infinity, the familiar model of σ-shift automorphism appears. These characteristics might help us understand the essential nature of chaotic spectra.

In one-dimensional maps the universality and scaling of period-η-tupling sequences are concisely described by the η-multifurcation functional renormalisation group equation

$$\varphi^{\eta}(\lambda_\eta x) = -\lambda_\eta \varphi(x) \qquad \varphi(0) = 1 \qquad \varphi'(0) = 1. \tag{1}$$

Here η is an arbitrary prime. It is very easy to extend η to any integer n; only some indices, such as ρ, need to be changed, i.e. $\rho = (n-1)/2$ (or $n/2$) when using any odd (or even) integer n for the period. $\circ$ indicates the composition of functions and φ is a universal invariant function. The universal scaling ratio for a period-η-tupling sequence is $\lambda_\eta = \alpha_\eta^{-1} = -\varphi^{\eta-1}(1)$. We can define a map based on (1). We have the orbit families $\{\varphi^j(x) | \varphi^i = \varphi^{i-1} \circ \varphi, \varphi^\circ = x, j \in N\}$ for any point $x \in (-1, 1) = I$. We have that for some initial point $x_0 \in \Omega \subset I$ (Ω is a non-wandering set), orbit families $\{\varphi^j(x_0)\}$ are almost periodic. In addition, we note that the measure of the set Ω on the I interval is non-zero (Jakobsen 1981). We can choose $x_0 = 0$ and give the representation of the line spectrum (Nauenberg and Rudnick 1981, Collet *et al* 1981) of map families in the frequency regions:

$$\tilde{\Phi}(\omega) = \sum_{k=-\infty}^{+\infty} a(q)\delta(\omega - k\omega_0) \tag{2}$$

$$a(q) = \frac{1}{p}\sum_{j=1}^{p-1} \varphi^j(0)\,\mathrm{e}^{\mathrm{i}2\pi kj/p} \tag{3}$$

where $q = k/p$, $p = p_N := \eta^N$, $\omega_0 = p^{-1}$, $k = 1, 2, \ldots, \eta - 1 (\mathrm{mod}\ \eta)$ and $\mathrm{i} = \sqrt{-1}$.

When p is large enough, since $\{\varphi^j(0)\}$ is almost periodic, the limit (2) and (3) exists. For generality, we will discuss the averaged square amplitude A_N^ν at level N:

$$\begin{aligned} A_N^\nu &= \eta^{1-N} \sum_{k=0}^{N-1} |a[(\eta k + \nu)\eta^{-N}]|^2 \qquad\qquad \sum_{k=0}^{N-1} \equiv \sum_{k=0}^{k=\eta^{N-1}-1} \\ &= \left(\frac{1}{p}\right)^2 \sum_{j=0}^{N-1} \left(\sum_{m=0}^{\eta-1} (b_m^j)^2 + 2 \sum_{m=0}^{\eta-2} \sum_{m'=m+1}^{\eta-1} e_{mm'}^\nu b_m^j b_{m'}^j \right) \quad (m \neq m') \end{aligned} \tag{4}$$

$$e_{m,m'}^\nu = \cos[2\pi(m-m')\nu/\eta] \qquad b_m^j = \varphi^{j+m\eta^{N-1}}(0) \qquad \nu = 0, 1, \ldots, \eta - 1. \tag{5}$$

We note that here we have the strongly phase coupled factors $\exp(-\mathrm{i}2\pi m\nu/\eta)$ among each amplitude b_m^j for k (mod η). But these phase factors are precisely η-primitive roots of 1, which form the independent unit system of the ηth real cyclotomic field.

According to the analysis by Feigenbaum (1979, 1980) a scaling function relation should exist in the difference of amplitudes between two successive orbits, defined as $\psi_{m,m'}^j = b_m^j - b_{m'}^j$. By using this identity, (5) can be rewritten as for $\nu = 1, 2, \ldots, \eta - 1$:

$$A_N^\nu = \left(\frac{1}{p}\right)^2 \sum_{j=0}^{N-1} \sum_{m=0}^{\eta-2} \sum_{m'=m+1}^{\eta-1} (-e_{m,m'}^\nu)(\psi_{m,m'}^j)^2 \tag{6}$$

and, for $\nu = 0$,

$$A_N^0 = \left(\frac{1}{p}\right)^2 \sum_{j=0}^{N-1} \left(\sum_{m=0}^{\eta-1} b_m^j \right)^2. \tag{7}$$

Note that $\max_{m,m',\nu}(-e_{m,m'}^\nu) = 1$. Obviously, for the power spectrum with different subcomponents ν at the same level N we have from (6) and (7) theorem 1.

Theorem 1. The main subcomponent ($\nu = 0$) of the power spectrum is larger than any other subcomponents ($\nu \neq 0 \bmod \eta$), namely $A_N^0 > A_N^\nu$.

This result is supported by all experimental results (Linsay 1981, Wang *et al* 1984) which have been observed. Other subcomponents ($\nu \neq 0$) at the same level N are dominated by the phase factors $e_{m,m'}^\nu$. They have rigorous symmetry and cyclometric properties.

From (6) we obtain

$$A_N^\nu = \sum_{i=1}^{\rho} (-e_{0,i}^\nu)\mathscr{D}_N^i \qquad (1 \leqslant \nu \leqslant (\eta-1)/2 =: \rho) \tag{8}$$

$$\mathscr{D}_N^i = \left(\frac{1}{p}\right)^2 \sum_{j=0}^{N-1} \left(\sum_{m=0}^{\eta-1-i} (\psi_{m,m+i}^j)^2 + \sum_{m=0}^{i-1} (\psi_{m,m+\eta-i}^j)^2 \right). \tag{9}$$

All subcomponents $\nu = 1, 2, \ldots, \eta - 1$ have the symmetry $A_N^{\nu} = A_N^{\eta-\nu}$ and satisfy

$$(A'_N \ldots A^{\rho}_N)^{\mathrm{T}} = \mu(\mathscr{D}'_N \ldots \mathscr{D}^{\rho}_N)^{\mathrm{T}}$$

where T stands for the transpose and μ is a matrix formed by the elements $e^{\nu}_{0,i}$ which belong to the ηth real cyclomotic field.

If we define the ratio between the squared and the crossed power spectrum as β,

$$\beta = \left(\sum_{m=0}^{\eta-2} \sum_{m'=m+1}^{\eta-1} b^{j}_{m} b^{j}_{m'}\right)\left(\sum_{m=0}^{\eta-1} (b^{j}_{m})^2\right)^{-1}$$

we have theorem 2 from (6) and (7), as follows.

Theorem 2. There is a restriction relation between the main subcomponent A^0_N and the averaged subcomponent spectrum over each $\nu(\nu \neq 0)$, i.e. $\langle A^{\nu}_N \rangle$:

$$A^0_N = \frac{\rho(1+2\beta)}{(\rho-\beta)} \langle A^{\nu}_N \rangle := \frac{\rho(1+2\beta)}{(\rho-\beta)} \frac{1}{\eta-1} \sum_{\nu=1}^{\eta-1} A^{\nu}_N.$$

From the constant positive property of the power spectrum we know that β will be confined by the inequality $(\eta-1)/2 > \beta > -\frac{1}{2}$.

For the same order η the spectrum ratio of the solution function for the renormalisation group equation at different levels N can be estimated. Without loss of generality, we discuss the ratio of the averaged peak height of the spectrum between the Kth and Mth levels:

$$A^{\nu}_N = \sum_{m=0}^{\eta-2} \sum_{m'=m+1}^{\eta-1} (-e^{\nu}_{m,m'}) \mathcal{E}^{m,m'}_N \qquad \mathcal{E}^{m,m'}_N = \left(\frac{1}{p_N}\right)^2 \sum_{j=0}^{N-1} (\psi^{j}_{m,m'})^2$$

$$\mathscr{D}^{i}_N = \sum_{m=0}^{\eta-1-i} \mathcal{E}^{m,m+i}_N + \sum_{m=0}^{i-1} \mathcal{E}^{m,m+\eta-i}_N.$$

Let K and M be fixed with $K > M$. We have

$$\mathcal{E}^{m,m'}_K = \left(\frac{1}{p_K}\right)^2 \sum_{k=0}^{M} \sum_{l=0}^{K-M-1} \left[\left(\frac{\mathrm{d}}{\mathrm{d}x}\varphi^k\right)(Z^{K,M}_{k,l,m,m'})(\varphi^{l\eta^M+m\eta^{K-1}}(0) - \varphi^{l\eta^M+m'\eta^{K-1}}(0))\right]^2$$

$$\begin{aligned} Z^{K,M}_{k,l,m,m'} &\in [\varphi^{l\eta^M+m\eta^{K-1}}(0); \varphi^{l\eta^M+m'\eta^{K-1}}(0)] \\ &= [(-\lambda_\eta)^M \varphi^{l+m\eta^{K-M-1}}(0); (-\lambda_\eta)^M \varphi^{l+m'\eta^{K-M-1}}(0)] \end{aligned} \qquad (10)$$

where $[a; b]$ denotes the closed interval with endpoints a, b. Using lemmas 1-3 for bounds on a continuous function (see the appendix) we can bound (10) for the same m and m':

$$(\lambda_\eta/\eta)^{2M} \delta_M \mathcal{E}^{m,m'}_{K-M} \leq \mathcal{E}^{m,m'}_K \leq (\lambda_\eta/\eta)^{2M} D_M \mathcal{E}^{m,m'}_{K-M}.$$

For the upper bound

$$D_M = \sum_{k=0}^{M} \sup_{l,m,m'} \left|\left(\frac{\mathrm{d}}{\mathrm{d}x}\varphi^k\right)(Z^{K,M}_{k,l,m,m'})\right|^2 \leq \sum_{k=0}^{M} \sup_{t\in[-1,1]} \left|\left(\frac{\mathrm{d}}{\mathrm{d}x}\varphi^k\right)(\lambda^M_\eta t)\right|^2 =: \Delta'_M. \qquad (11)$$

For the lower bound we have, from lemma 3,

$$\delta_M = \sum_{k=0}^{M} \inf_{t\in[-1,1]} \left|\left(\frac{\mathrm{d}}{\mathrm{d}x}\varphi^k\right)(\lambda^M_\eta t)\right|^2 = 1.$$

In order to estimate the value of Δ'_M we need to discuss the Kth compound derivative ($1 \leqslant k \leqslant \eta^M$). We can decompose an arbitrary integer k into a sequence of prime η consisting of its sum and product. We know from number theory that the following η decomposition is valid and one sequence of indices ν_j, σ_q, corresponding to one integer, is unique in a η-adic system or in lexicographic order:

$$k = \sum_{j=0}^{r-1} \nu_j \eta^{\sum_{q=0}^{j} \sigma_q}.$$

Here $\sigma_0 \geqslant 0 (q=0)$, $\sigma_q \geqslant 1 (q>1)$, $\nu_j = 1, 2, \ldots, \eta-1 (\forall j, j=0,1,\ldots,r-1)$ and they are all positive integers. When k is fixed, each ν_j takes only one value and $r=r(k)$ satisfies the inequality $r \leqslant M - \max_q \sigma_q + 1$. We obtain

$$\left|\left(\frac{\mathrm{d}}{\mathrm{d}x}\varphi^k\right)(\lambda_\eta^M t)\right| = \lambda_\eta^{-M}\left|\frac{\mathrm{d}}{\mathrm{d}t}\lambda_\eta^{\sigma_0}\varphi^{\nu_0}(\lambda_\eta^{\sigma_1}\varphi^{\nu_1}\ldots(\lambda_\eta^{\sigma_{r-1}}\varphi^{\nu_{r-1}}(\lambda_\eta^{\sigma_r}t)\ldots))\right|.$$

Here $\sigma_r = M - (\sigma_0 + \ldots + \sigma_{r-1}) \geqslant 1$. By the chain rule for differentiation we find a more complex expression than in the case of $\eta = 2$:

$$\left|\left(\frac{\mathrm{d}}{\mathrm{d}x}\varphi^k\right)(\lambda_\eta^M t)\right| = \prod_{j=1}^{r}\prod_{\mu=0}^{\nu_{j-1}-1}\varphi'(\varphi^\mu(\lambda_\eta^{\sigma_j}\varphi^{\nu_j}(\lambda_\eta^{\sigma_{j+1}}u_{r-j-1}))) =: \prod_{j=1}^{r}\prod_{\mu=0}^{\nu_{j-1}-1} {}^\mu A_{\sigma_j}^{\nu_{j-1}}$$

$$u_j = \varphi^{\nu_{r-j}}(\lambda_\eta^{\sigma_{r-j+1}}u_{j-1}) \qquad u_0 = t \qquad B_{\sigma_j}^{\nu_{j-1}} = \prod_{\mu=0}^{\nu_{j-1}-1} {}^\mu A_{\sigma_j}^{\nu_{j-1}} a_{\sigma_j}^{-1}. \tag{12}$$

Here, using lemmas 1-3,

$${}^0 A_{\sigma_j}^{\nu_{j-1}} = a_{\sigma_j} b_{\sigma_{j+1}}^{\nu_j} \qquad a_{\sigma_j} = (\lambda_\eta)^{\sigma_j} \qquad b_{r+1}^{\nu_r} := \gamma_M^2 \tag{13}$$

$$b_{\sigma_{j+1}}^{\nu_j} = \begin{cases} \gamma_M^2|\varphi^{\nu_j}(0)|^2 & \sigma_{j+1} > 1 \\ \gamma_M^2|\varphi^{\nu_j}(\lambda_\eta)|^2 & \sigma_{j+1} = 1 \end{cases} \qquad \nu_j = 1, 2, \ldots, \eta - 1 \tag{14}$$

$${}^\mu A_{\sigma_j}^{\nu_{j-1}} = \gamma_M^2|\varphi^\mu(\lambda_\eta^{\sigma_j})|^2 = \begin{cases} \gamma_M^2|\varphi^\mu(0)|^2 & \sigma_{j+1} > 1 \\ \gamma_M^2|\varphi^\mu(\lambda_\eta)|^2 & \sigma_{j+1} = 1 \end{cases} \qquad \mu = 1, 2, \ldots, \nu_{j-1} - 1. \tag{15}$$

To transform the old indices of the summation into new indices r, σ, ν we obtain

$$D_M \leqslant \Delta'_M \leqslant 1 + \max_{\nu,\sigma}(\gamma_M^2(b_\sigma^\nu)^{-1}) \sum_{\sigma_0=0}^{M-1} \lambda_\eta^{2(M-\sigma_0)} \sum_{r=1}^{M-\sigma_0}$$

$$\times \sum_{\substack{\sigma_1+\ldots+\sigma_r = M-\sigma_0 \\ \sigma_i \geqslant 1, \forall i}} \sum_{\substack{r \leqslant \nu_0+\ldots+\nu_{r-1} \leqslant (\eta-1)r \\ \nu_i = 1,2,\ldots,(\eta-1), \forall i}} \prod_{j=1}^{r} B_{\sigma_j}^{\nu_{j-1}} =: \Delta_M. \tag{16}$$

In a similar way to Collet *et al* (1981), the method of generating functions can be generalised. In order to evaluate Δ_M we introduce

$$C_L = \sum_{r=1}^{L} \sum_{\substack{\sigma_1+\ldots+\sigma_r = M-\sigma_0 \\ \sigma_j \geqslant 1, \forall j}} \sum_{\substack{r \leqslant \nu_0+\ldots+\nu_{r-1} \leqslant (\eta-1)r \\ \nu_i = 1,2,\ldots,(\eta-1), \forall i}} \prod_{j=1}^{r} B_{\sigma_j}^{\nu_{j-1}} \qquad L > 0 \tag{17}$$

where $C_0 := 1$ and $C_L = 0$ for $L < 0$. We obtain the following recurrence relation from (17):

$$C_L = \sum_{\nu=1}^{\eta-1} \sum_{\sigma=1}^{L} B_\sigma^\nu C_{L-\sigma}. \tag{18}$$

From the method of generating functions and (12)-(15) we have

$$F(x)=\sum_{j=0} C_j x^j \qquad C_L=\alpha_+ x_+^{-L-1}+\alpha_- x_-^{-L-1}$$
$$\alpha_\pm=(x_\pm-1)/(x_\pm-x_\mp)(B_1-B_2) \qquad B_6=\sum_{\nu=1}^{\eta-1} B_\sigma^\nu. \tag{19}$$

Substituting back into (16) we obtain

$$D_M\leqslant \Delta_M=1+\max_{\nu,\sigma}[\gamma_M^2(b_\sigma^\nu)^{-1}]\sum_{\sigma_0=0}^{M-1}\lambda_\eta^{2(M-\sigma_0)}C_{M-\sigma_0}$$
$$\leqslant 1+\frac{\lambda_\eta^2}{\max_{\nu,\sigma}(\gamma_m^{-2}b_\sigma^\nu)}\left(\frac{\alpha_+(\beta_+^M-1)}{x_+^2(\beta_+-1)}+\frac{\alpha_-(\beta_-^M-1)}{x_-^2(\beta_-^M-1)}\right)\leqslant EF^M \qquad \beta_\pm=\frac{\lambda_\eta^2}{x_\pm}. \tag{20}$$

Theorem 3. For all $M>0$, the averaged power spectrum over each subcomponent ν of the η-multifurcation renormalisation group equation (1) has the lower and upper bound:

$$(\lambda_\eta/\eta)^{2M}\leqslant\langle A_{N+M}^\nu\rangle/\langle A_N^\nu\rangle\leqslant(\lambda_\eta/\eta)^{2M}EF^M. \tag{21}$$

Corollary. The two successive averaged square amplitudes, i.e. $M=1$, have the asymptotical bounds

$$(\lambda_\eta/\eta)^2\leqslant\langle A_{N+1}^\nu\rangle/\langle A_N^\nu\rangle=:\langle\mu_N\rangle\leqslant(\lambda_\eta/\eta)^2D_1.$$

The asymptotic decrease of two successive square amplitudes is restricted by the inequality

$$f_{\min}<-10\log\langle\mu_\eta\rangle<f_{\max}. \tag{22}$$

We note that this estimation of bounds for the same level N is independent of m and m'. This shows the scaling property of ψ. The approximate numerical solution of the renormalisation group equation of order n ($n=3,4,5$) is known from Zeng *et al* (1984). We can calculate the behaviour of φ at the points $x=0$ or λ and obtain the values of E and F from (12)-(20). We list these values of E, F, D_1, $f_{\min}$ and $f_{\max}$ (for $n=3,4,5$) in table 1.

Table 1. Calculated values of the spectrum bounds for period-n-tupling sequences.

n Sequence	2^a $(R)^{*n}$	3 $(RL)^{*n}$	4 $(RL^2)^{*n}$	5 $(RLR^2)^{*n}$	5 $(RL^2R)^{*n}$	5 $(RL^3)^{*n}$
α_η [b]	2.5029	9.2773	38.819	20.128	−45.804	160.0
E	13.996	9.3074	10.391	28.467	36.319	13.225
F	1.1088	1.2682	1.2389	1.2819	1.2455	1.1783
D_1	2.493	3.4962	3.4828	9.0261	9.9178	3.3587
$f_{\min}$	10.022	23.455	38.403	30.50	37.233	52.80
$f_{\max}$	13.989	28.891	43.822	40.055	47.197	58.06

[a] This result is taken from Collet *et al* (1981) and is the same as in this paper.
[b] See Zeng *et al* (1984).

Here we consider the restraint bound of the power spectrum for each subcomponent. Because of coupled phase factors among them, the case is slightly more complex than the averaged spectrum. We rewrite (8) and divide the phase factors into both positive, $-e^{\nu}_{0,i}>0$ $(i\in i_+)$, and negative, $-e^{\nu}_{0,i}<0 (i\in i_-)$, parts and note the inverse of the matrix μ

$$\mu^{-1}=\{h^{j}_{0i}\} \qquad \mathscr{D}^{i}_{k}=\sum_{j=1}^{\rho} h^{j}_{0i}A^{j}_{K}.$$

We finally obtain the restraint relation of the power spectrum for each subcomponent, as follows:

$$\sum_{j=1}^{\rho} u^{\nu}_{j}A^{j}_{K-M}<A^{\nu}_{K}<\sum_{j=1}^{\rho} v^{\nu}_{j}A^{j}_{K-M} \qquad (\nu=1,2,\ldots,\rho)$$

$$u^{\nu}_{j}=(\lambda_{\eta}/\eta)^{2M}\left(EF^{M}\sum_{i_-}(-e^{\nu}_{0i})(h^{j}_{0i})+\sum_{i_+}(-e^{\nu}_{0i})(h^{j}_{0i})\right)$$

$$v^{\nu}_{j}=(\lambda_{\eta}/\eta)^{2M}\left(\sum_{i_-}(-e^{\nu}_{0i})(h^{j}_{0i})+EF^{M}\sum_{i_+}(-e^{\nu}_{0i})(h^{j}_{0i})\right).$$

This result shows that, although the local difference of amplitudes has scaling independent of m, m', the global power spectrum of each subcomponent A^{ν}_{N} is naturally modulated by the phase factors $(-e^{\nu}_{0i})$. A similar case also exists in the multifurcation sequences of higher periodic orbits in non-linear mechanical systems when we use the renormalisation group method to analyse them (Peng *et al* 1985).

We have given rigorous bounds for the scaling factors for the power spectrum of period-η-tupling sequences, but the numerical values of these bounds necessarily depend on the numerical solution of (1) (Zeng *et al* 1984) and on a prior estimate of the behaviour of the function in the interval $[-1, 1]$. At present, an exact description of the scaling factors for the power spectrum of period-η-tupling sequences is, to our knowledge, still a question which has not yet been solved. We believe that the above conclusions are useful in the study of dissipative systems in which multifurcation phenomena occur.

Appendix

In order to estimate the upper bound for the power spectrum we confine the φ to an appropriate solution class of the renormalisation group equation by reasonable assumptions. This is that the $\varphi(x)$ satisfying equation (1) is continuous on the interval $[-1, 1]$, and it belongs to the function class C^3 $(\varphi\in C^3)$ and is concave and even (Campanino and Epstein 1981, Epstein and Lascoux 1981). From this assumption we can write several lemmas on the bound of the function φ as follows. Because these lemmas are very simple we omit the proof.

Lemma 1. If φ satisfies equation (1) when $x\in[-1,1]$, $0<\lambda<1$, then $\varphi\in[-1,1]$ and $\varphi^m\in[-1,1]$, for all positive integer m.

Corollary. When $0<\lambda<1$ and $t\in[-1,1]$, we obtain $\|u_i\|\leq 1$, $\|\varphi^{\nu}(\lambda^{\sigma}u_i)\|\leq 1$ in the former proof of theorem 3.

Lemma 2. If $\varphi \in C^3$ and φ is concave and even then its second derivative has negative bounds whose absolute values are

$$\gamma_M = \sup_{x\in[-1,1]} |\varphi''(x)| = -\varphi''(0) \qquad \gamma_m = \inf_{x\in[-1,1]} |\varphi''(x)| = -\varphi''(1).$$

Lemma 3. If the function φ is concave and even, then the bound of the first derivative $\varphi_j' = \varphi'(\lambda^{\sigma_j} u_j)$ is $0 \leq |\varphi_j'| \leq \gamma_M \lambda^{\sigma_j}$.

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